The Effect of the Light Round-Trip Time on the Performance of an Adaptive Optics Turbulence Compensation System

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1. Introduction.

The propagation of a laser beam through atmospheric turbulence can, under many circumstances, be compensated to near diffraction limited levels, by use of an adaptive optics system. In the ideal case, a beacon light source is generated at the aimpoint. This light traverses through the turbulent path back to the aperture. The distribution of phase perturbations in the aperture plane is sensed, and the reverse of this phase is applied to the outgoing beam. When the turbulent path is long (e.g. a few hundred kilometers), and the effective wind speed is high (such as is the case when the laser is based on a plane, with airspeeds around 200 m/s), the performance of the adaptive optics system can be significantly degraded due to the time delay from when the beacon samples the turbulence, until the beam propagates through the turbulence. This effect has not been treated in the past, because for astronomical applications, the time delays due to the round trip time of light are too short (tens of microseconds) to cause problems.

This paper presents an analysis of the anisoplanatic effect of the round trip time of light. A formulation of the phase variance has been constructed for this effect. The time response of the adaptive optics system (i.e. servo bandwidth) is intimately related to the light round trip time effect, and is included in the analysis. Results are first obtained for the large aperture limit, and then the effects of finite aperture size are treated. A set of scaling relationships have been found that enable the formulation to be cast in a form that is insensitive to the turbulence strength profile along the propagation path.

2. Phase Variance Formulation.

A formulation for the Strehl ratio for a focused compensated beam propagating through turbulence, was developed in a parent paper [Stroud,1992]. That formulation can be applied to the anisoplanatic effect caused by the finite speed of light. In the case that amplitude perturbations are negligible and the Strehls are greater than about 0.3 (the Marechal limit), the Strehl was given by

$$S \equiv \frac{< I >}{I_{DL}} = \exp\left(- \int_A <\varphi_A^2> + < \int_A \int_B \varphi_A \varphi_B >\right) \equiv \exp(-\sigma_\varphi^2) \qquad [1]$$

φ_A is the phase error at position $\rho_\mathbf{A}$ in the aperture plane. It is the phase perturbation due to turbulence for light propagating from $\rho_\mathbf{A}$ to the aimpoint, minus the phase correction applied at $\rho_{\mathbf{A}}$. The average over the aperture is designated by $\int_A$. The phase variance is the negative of the argument of the exponential, i.e. σ_φ^2.

For a perfect sensor and deformable mirror, using a beacon source located at the aimpoint, the applied phase correction will be the phase increment sensed in the beacon light arriving at $\rho_{\mathbf{A}}$. The effect that will be treated here is the delay between the time the beacon samples a given turbule and the time the outgoing beam propagates through the turbule. During this delay, the turbule will move due to an effective perpendicular wind, $\mathbf{w}(z)$. The outgoing beam will thus propagate through a different turbulence field than the beacon light. This delay will have two components. There is the round trip time of light, given by twice the distance from the aperture to the turbule divided by the speed of light. There will also be a delay due to the finite time between receipt of the beacon signal and emission of the compensated beam. This second delay represents the combined effects of a finite beacon sensor frame time, a finite processing time, and/or a finite bandwidth AO control loop for the deformable mirror.

In the notation of the previous paper, the phase perturbation received at time 0, at aperture position $\rho_{\mathbf{A}}$, is given by

$$\varphi_A^{IN}(0)=k_o \int_0^L dz\sqrt{0.066C_n^2(z_o)}\int\frac{d^3\vec{k}}{k^{11/6}}\Big[\cos(\vec{k}\quad\vec{\rho}_a t/L + k_{ll}(z-z_o)-\vec{k}\quad\vec{w}z/c+\varphi(\vec{k},z_o)\Big] \quad [2]$$

while the phase perturbation on the outgoing light, emitted at time τ, is given by

$$\varphi_A^{OUT}(\tau)=$$

$$k_o\int_0^L dz\sqrt{0.066C_n^2(z_o)}\int\frac{d^3\vec{k}}{k^{11/6}}\Big[\cos(\vec{k}\quad\vec{\rho}_a t/L + k_{ll}(z-z_o)+\vec{k}\quad\vec{w}(z/c+\tau)+\varphi(\vec{k},z_o)\Big] \quad [3]$$

k_o is the laser wavenumber in vacuum, $2\pi/\lambda$. z is the distance along the propagation path, where z=0 at the aperture, and z=L at the aimpoint. z_o is an arbitrary point in the vicinity of z. C_n^2 is the index of refraction structure coefficient, which characterizes the strength of the turbulence. By definition, the expected value of the square of the difference of index of refraction at two points separated by a distance $\mathbf{r}$ is $<(n(\mathbf{r}+\mathbf{r'})-n(\mathbf{r'}))^2> = C_n^2\ r^{2/3}$. $\mathbf{k}$ is the wavenumber of Fourier cosine expansion component of the turbulence. t is L-z. $\mathbf{w}$ is the effective wind speed as a function of z, with which the turbulence is blowing across the path. $\phi(\mathbf{k}, z_o)$ is the phase of the turbulence component, which is a uniformly distributed random variable.

The compensated phase error, obtained by subtracting the phase perturbation received at time 0 from the phase perturbation of the outgoing beam emitted at time τ, is then

$$\varphi_A(\tau)=k_o\int_0^L dz\sqrt{0.066C_n^2(z_o)}\int\frac{d^3\vec{k}}{k^{11/6}}$$

$$\{\Big[\cos(\vec{k}\quad\vec{\rho}_a t/L + k_{ll}(z-z_o)-\vec{k}\quad\vec{w}z/c+\varphi(\vec{k},z_o)\Big]$$

$$-\Big[\cos(\vec{k}\quad\vec{\rho}_a t/L + k_{ll}(z-z_o)+\vec{k}\quad\vec{w}z/c+\vec{k}\quad\vec{w}\tau+\varphi(\vec{k},z_o)\Big]\} \qquad [4]$$

Following the development presented in the previous paper, the phase variance is found to be

$$\sigma_\varphi^2 = 1.303\, k_o^2 \int_0^L dz\ C_n^2(z) \int \frac{dk}{k^{8/3}} \left\{ 1 - \left[\frac{2\, J_1(k\ Rt/L)}{(k\ Rt/L)} \right]^2 \right\} \left[2 - 2\, J_0\left(k\ w(\tau + 2z/c)\right) \right] \quad [5]$$

In the limit of large aperture, the J_1 term of Eq[5] goes to zero. The phase variance is then

$$\lim_{R\to\infty} \sigma_\varphi^2 = 1.303\, k_o^2 \int_0^L dz\ C_n^2(z) \int \frac{dk}{k^{8/3}} \left[2 - 2J_0\left(k\ w(\tau+2z/c)\right) \right]$$

$$= 2.914\, k_o^2 \int_0^L dz\ C_n^2(z)\ w^{5/3} \tau^{5/3} \left(1 + \frac{2z}{\tau c}\right)^{5/3} \quad [6]$$

3. Large Aperture, Infinite Light Speed, Finite Bandwidth Case.

In the limit of infinite light speed, this result is identical to formulations obtained by Tyler [Tyler,1989] and Greenwood [Greenwood,1977]. In this limit, the phase variance is

$$\lim_{R;c\to\infty} \sigma_\varphi^2 = (20.13\tau)^{5/3} \left[0.01956\, k_o^2 \int_0^L dz\ C_n^2(z)\ w^{5/3} \right]$$

$$= (20.13\tau)^{5/3} f_o^{5/3} \quad [7]$$

where f_o is a frequency characterizing the effect of wind, given by Greenwood as

$$f_o = \left[0.01956\, k_o^2 \int_0^L dz\ C_n^2(z)\ w^{5/3} \right]^{3/5} \quad [8]$$

The delay time, τ, can thus be interpreted in terms of the AO servo bandwidth. For an AO system which perfectly compensates all temporal fluctuations with frequencies below f_c, but doesn't respond to higher frequencies (i.e. a bandpass filter with a sharp cut-off at frequency f_c), the response is equivalent to using a delay time of $1/(20.13 f_c)$. For an RC filter, with 3dB bandwidth of f_{3dB} (i.e. a bandpass filter $(f/f_{3dB})^2/[1+(f/f_{3dB})^2]$), the response is equivalent to using a delay time of $2.69/(20.13 f_{3dB})$. With these interpretations of the delay time, the phase variance (with infinite light speed, large aperture, and negligible scintillation) is

$$\sigma_\varphi^2 = (f_o / f_c)^{5/3} = (2.69\, f_o / f_{3dB})^{5/3} \quad [9]$$

The quantity $2.69\, f_o$ is also known as the Greenwood frequency.

4. Large Aperture, Finite Light Speed, Infinite Bandwidth Case.

In the limit of large aperture and zero delay time (i.e. infinite AO servo bandwidth), the phase variance including the finite speed of light is given by

$$\lim_{R\to\infty} \sigma_\varphi^2 = 2.91\, k_o^2 \int_0^L dz\ C_n^2(z)\ w^{5/3}\, (2z/c)^{5/3}$$

$$= (20.13\ \ 2L/c)^{5/3}\, f_s^{5/3}$$

$$= (f_s / f_L)^{5/3} \qquad [10]$$

where a new characteristic frequency, f_s, is defined by

$$f_s = \left[\, 0.01956\, k_o^2 \int_0^L dz\ C_n^2(z)\, (wz/L)^{5/3}\right]^{3/5} \qquad [11]$$

and the light round trip time bandwidth is defined by

$$f_L = 1/(20.13 * 2L/c). \qquad [12]$$

As an example, for the Hufnagel-Valley 5/7 turbulence profile, using a light wavelength of 2.314 µm, propagating from an altitude of 41kft to an aimpoint located at a cross-range of 400 km and an altitude of 25 km, with the effective wind speed given by (200 m/s)(L-z)/L, this frequency is found to be 17.96 Hz. This gives a phase variance (including piston) of 0.941 rad^2, corresponding to a Strehl of 0.39. This phase variance scales with wavelength to the minus six fifths power.

5. Large Aperture, Finite Light Speed, Finite Bandwidth Case.

For the large aperture limit, Eq[6] can be recast into a normalized phase variance:

$$y = \lim_{R\to\infty} \frac{\sigma_\varphi^2}{(f_s/f_L)^{5/3}} = \frac{\int_0^1 d\gamma\ C_n^2(z\gamma)\ \ w^{5/3}\left(\frac{f_s}{f_o}\, x - \gamma\right)^{5/3}}{\int_0^1 d\gamma\ C_n^2(z\gamma)\ \ w^{5/3}\ \ \gamma^{5/3}} \qquad [13]$$

where $\gamma = z/L$, and

$$x \equiv \frac{\tau f_o c}{2L f_s} = \frac{f_o}{f_c}\frac{f_L}{f_s} = \text{normalized delay time} \qquad [14]$$

For practical purposes, this normalized phase variance is a function of the single parameter x. The dependence on the turbulence profile and the normal wind profile is incorporated into the frequencies f_o and f_s. To justify this astounding assertion, the normalized phase variance as a function of x has been evaluated with several turbulence profiles and for three wind profile cases.

For the first case, the turbulence profile is taken to be a delta function, i.e. $C_n^2(z) = \delta(z-z_o)\, C_n^2$. In this case, $f_s/f_o = z_o/L$, and the normalized phase variance is simply

$$y = \left(x + 1\right)^{5/3} \qquad [15]$$

This normalized phase variance is shown in Fig.1, as a function of the single parameter $1/x = \frac{f_c}{f_o}\frac{f_s}{f_L}$. For this profile, there is no dependance on the wind profile. When the AO bandwidth, represented by f_c, is large, the phase variance approaches the value given by Eq[10], and the normalized phase variance approaches unity. When the bandwidth is small, the phase variance approaches the value given by Eq[9].

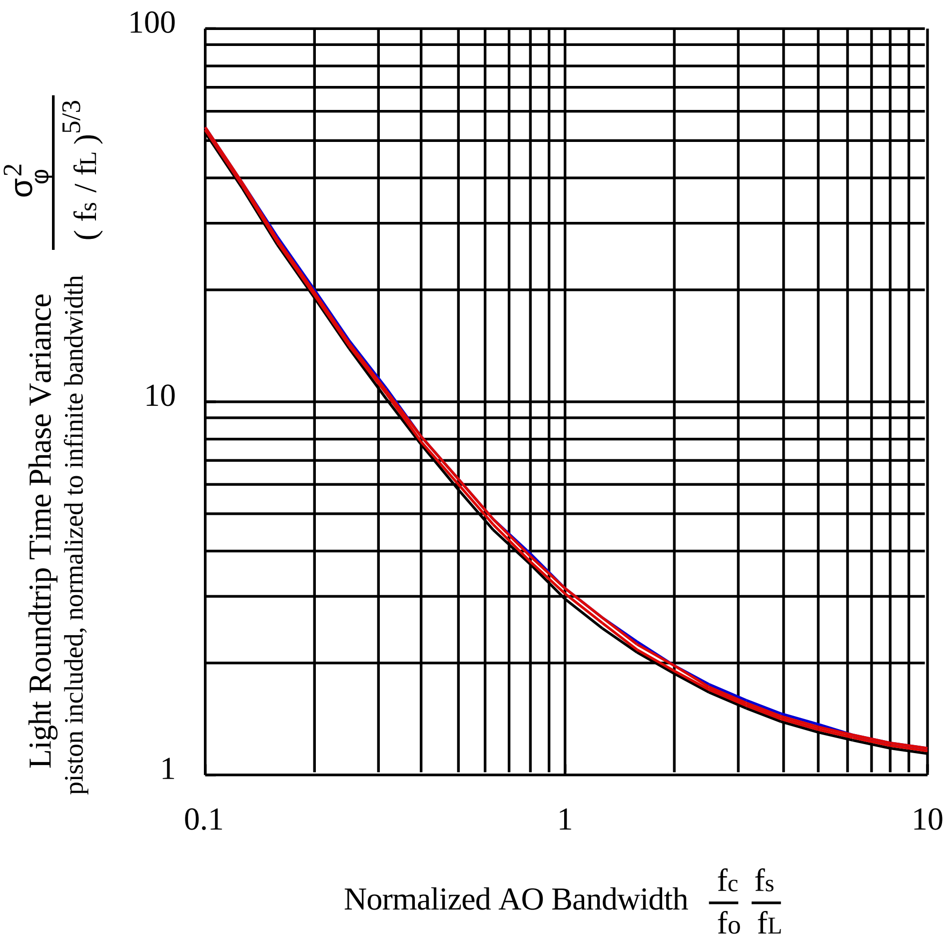

Figure 1. The normalized phase variance (including piston) including the finite speed of light and the AO bandwidth. The phase variance is normalized to the infinite bandwidth phase variance. The four cases shown are in ascending order: uniform turbulence profile with effective wind due to motion of aperture, uniform turbulence profile with uniform wind, uniform turbulence profile with effective wind due to motion of target, delta function turbulence profile.

For the second case, a uniform turbulence profile is taken. $C_n^2(z)$ is constant along the path and can be taken outside the z integral. Now if the effective wind is due to the motion of the laser, i.e. w= ~ (1-γ), then f_s/f_o = 0.31992, and the normalized phase variance is

$$y = 17.7268 \int_0^1 d\gamma\,(1-\gamma)^{5/3}\left(0.31992\,x + \gamma\right)^{5/3} \qquad [16]$$

If the effective wind is constant along the propagation path, there follows

$$y = \left(1 + 0.555161\,x\right)^{8/3} - \left(0.555161\,x\right)^{8/3}. \qquad [17]$$

Finally, when the effective wind is due to slewing of the target, i.e. $w \sim \gamma$, then

$$y = 4.33333 \int_0^1 d\gamma\;\gamma^{5/3}\left(0.747288\,x + \gamma\right)^{5/3} \qquad [18]$$

The normalized phase variances obtained from Eqs[16,17 &18] are also shown in Fig.1, again as a functions of the single parameter, $1/x = \frac{f_c\;f_s}{f_o\;f_L}$. The relation shown in Fig.1 has been found to hold for a variety of turbulence and wind profiles.

6. Finite Aperture, Finite Light Speed, Infinite Bandwidth Case.

When the aperture size is finite, the J_1 term of Eq[5] must be retained. The case of sufficiently large bandwidth will be treated, so that τ=0. With a delta function turbulence profile, the normalized phase variance with piston removed will be

$$y = 0.8942 \int \frac{d\mu}{\mu^{8/3}} \left\{1 - \left[\frac{2J_1(\mu\xi)}{\mu\xi}\right]^2\right\} \left[1 - J_0(\mu)\right] \qquad [19]$$

$$= K(\xi)$$

where $\xi \equiv \frac{1.59\;D\;c}{20.13\;f_s\;r_0\;2L}$ is a normalized aperture, and r_0 is Fried's atmospheric coherence length for the path from 0 to L, defined by [Fried,1966]:

$$r_0 \equiv \left[\frac{2.91}{6.88}\,k_o^2 \int_0^L dz\;C_n^2(z)(1-z/L)^{5/3}\right]^{-3/5} \qquad [20]$$

The normalized phase variance obtained by numerical integration of Eq[19], is shown in Fig.2, expressed as a function of the normalized aperture.

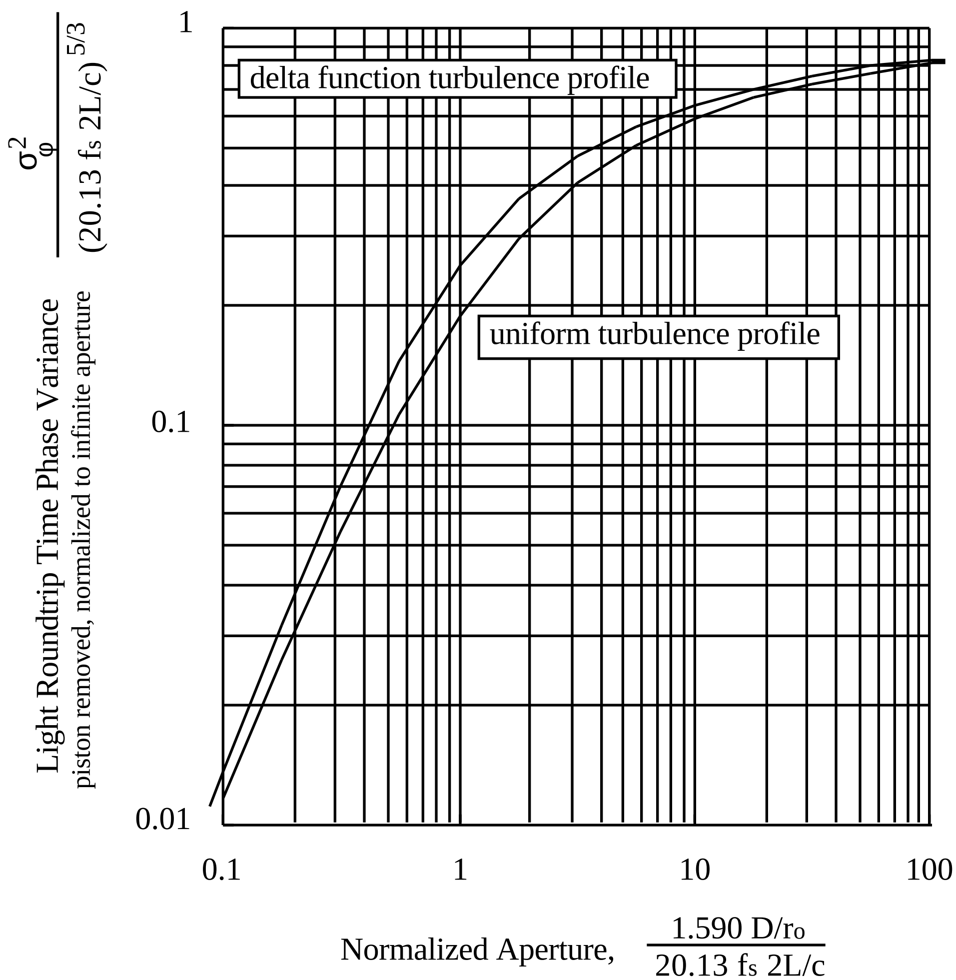

Figure 2. The normalized light round trip time phase variance (piston removed) as a function of a normalized aperture. The piston removed phase variance is normalized to the piston included phase variance, i.e. the large aperture phase variance. The aperture is normalized to the atmospheric coherence length times the characteristic frequency defined in the text times the light round trip time.

In the limit of small normalized aperture, the phase variance approaches the asymptotic value of

$$\sigma_\varphi^2 = 2.066\,(D / r_0)^{5/3} \quad \text{for small } \frac{1.59\ D\ c}{20.13\ f_s\ r_0\ 2L} \qquad [21]$$

This phase variance is exactly twice the phase variance of uncompensated turbulence. When the normalized aperture is sufficiently small, the phase error sensed by the beacon is uncorrelated to the required phase correction of the outgoing beam, because the turbulence changes too much during the light round trip time. The total variance is then the sum of the beacon and outgoing beam variances, i.e. twice the uncompensated result. When ξ=0.316, the compensated beam has the same phase variance as would be obtained with no compensation.

For the uniform turbulence profile, the normalized phase variance is given by

$$y = 17.7268 \int_0^1 d\gamma \, (1-\gamma)^{5/3} \, \gamma^{5/3} \, K\left(0.31992 \, \frac{\xi}{\gamma}\right) \qquad [22]$$

A numerical evaluation of the normalized phase variance is also shown in Fig.2 for this case.

When D/r_0 is specified, the phase variance becomes a function of the single parameter $20.13 \, f_s \, 2L/c$. In the example given before (Hufnagel-Valley 5/7 turbulence profile, wavelength of 2.314 µm, propagating from an altitude of 41kft to an aimpoint located at a cross-range of 400 km and an altitude of 25 km, plane speed of 200 m/s) the coherence length is 38.2 cm. For a 2 meter aperture, this gives $D/r_0 = 5.21$, and the normalized phase variance is 0.56. Since the large aperture limit phase variance was 0.941 rad^2, the phase variance with piston removed is then 0.527 rad^2. The Strehl is then 0.59 instead of 0.39. Fig.3 shows the Strehl as a function of $20.13 \, f_s \, 2L/c$, for several values of D/r_0, using a delta function profile formulation.

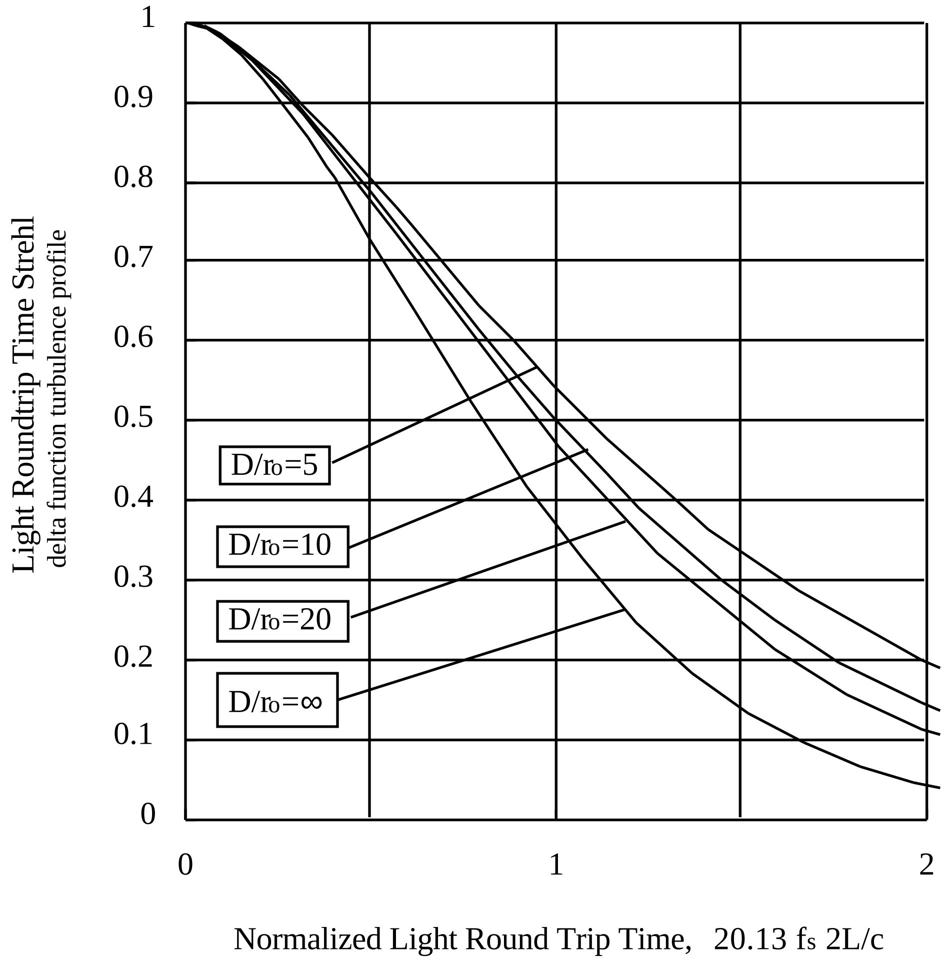

Figure 3. Light round trip Strehl as a function of a normalized light round trip time, for several ratios of aperture to atmospheric coherence length. The case shown is for the delta function class of turbulence profiles. The light round trip time is normalized with the characteristic frequency defined in the text.

7. Summary.

The anisoplanatic effect due to the finite speed of light, has been analyzed. It has been found that when the light round trip time is significant relative to a characteristic time, i.e. $1/(20.13f_s)$, the intensity of the beam at the aimpoint will be degraded from diffraction limited values. The beam degradation due to finite light speed was obtained for three cases: infinite bandwidth and large aperture; finite bandwidth and large aperture; and infinite bandwidth and finite aperture. Several turbulence strength profiles were treated, and a formulation was developed which is relatively insensitive to the details of the profile.

8. References.

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